COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 37, No. 2, 1988

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171 JAPAN

A Result on Coverings of $\mathbb{P}^n$ Branched along a Hypersurface with Only Simple Normal Crossings

by

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(Received January 18, 1988)

By a *branched covering*, we mean a finite surjective morphism of normal complex projective varieties. Lazarsfeld [7] showed that if a non-singular n-dimensional projective variety X admits a branched covering $X\to\mathbb{P}^n$ of degree d, then the induced homomorphisms $H^i(\mathbb{P}^n, \mathbb{C})\to H^i(X, \mathbb{C})$ are isomorphisms for $i\leqq n+1-d$. In this paper, we shall study somewhat different type of branched coverings of projective space and prove the following analogue of Lazarsfeld's theorem:

THEOREM. *Let $f: X\to\mathbb{P}^n$ be a branched covering which is étale over the complement of a hypersurface in $\mathbb{P}^n$ with only simple normal crossings. Then the induced homomorphisms*

$$f^*: H^i(\mathbb{P}^n, \mathbb{C}) \longrightarrow H^i(X, \mathbb{C})$$

are isomorphisms for $i\neq n$.

Notation. If $\mathscr{E}$ is a vector bundle and t is a nonnegative integer, $\mathscr{E}^{\otimes t}$ (resp. $S^t\mathscr{E}$, $\Lambda^t\mathscr{E}$) denotes the t-th tensor (resp. symmetric, exterior) power of $\mathscr{E}$. $\mathscr{E}^*$ denotes the dual bundle of $\mathscr{E}$. If $\mathscr{E}$ is a line bundle and t is a negative integer, $\mathscr{E}^{\otimes t}$ means $(\mathscr{E}^*)^{\otimes(-t)}$. If X is a non-singular variety, Ω_X^p denotes the sheaf of regular p-forms on X. Given a morphism $X\to Y$ of varieties, we denote by $\Omega_{X/Y}$ the sheaf of relative differentials of X over Y.

§1. Reduction of the problem

Let S be a hypersurface in $\mathbb{P}^n$ with only simple normal crossings and let $S_1, S_2, \cdots, S_m$ be the irreducible components of S. Then one has $\pi_1(\mathbb{P}^n-S)\cong \mathbb{Z}\alpha_1\oplus\mathbb{Z}\alpha_2\oplus\cdots\oplus\mathbb{Z}\alpha_m/(n_1\alpha_1+n_2\alpha_2+\cdots+n_m\alpha_m)$, where α_i is a small loop around S_i and n_i is the degree of S_i (cf. [8]). To each subgroup $\mathfrak{H}\subseteqq\pi_1(\mathbb{P}^n-S)$ of finite index, there corresponds one-to-one a branched covering $f: X\to\mathbb{P}^n$ of which the restriction $f|_{X-f^{-1}(S)}: X-f^{-1}(S)\to\mathbb{P}^n-S$ is the étale covering associated with $\mathfrak{H}$. For each positive integer ν, let $\mathfrak{H}_\nu$ be the subgroup of $\pi_1(\mathbb{P}^n-S)$ generated by the elements $\nu n_1\alpha_1, \nu n_2\alpha_2, \cdots, \nu n_m\alpha_m$. We call the branched covering $f_\nu: U_\nu\to\mathbb{P}^n$ which corresponds to $\mathfrak{H}_\nu$ the *ν-th Kawai covering* associated with the hypersurface S. For an arbitrary branched covering $f\colon X\to\mathbb{P}^n$ which is étale over $\mathbb{P}^n-S$, one can find a subgroup $\mathfrak{H}_\nu$ which is also a subgroup of the image $f_*\pi_1(X-f^{-1}(S))$. This implies that there is a finite surjective morphism $g\colon U_\nu\to X$ such that $f_\nu=f\circ g$:

$$\begin{tikzcd} U_\nu \arrow[rr, "g"] \arrow[dr, "f_\nu"'] & & X \arrow[dl, "f"] \\ & \mathbb{P}^n . & \end{tikzcd}$$

By Satz 7 in [4], we know that the induced homomorphisms $f_\nu^*\colon H^i(\mathbb{P}^n, \mathbb{C})\to H^i(U_\nu, \mathbb{C})$, $f^*\colon H^i(\mathbb{P}^n, \mathbb{C})\to H^i(X, \mathbb{C})$ and $g^*\colon H^i(X, \mathbb{C})\to H^i(U_\nu, \mathbb{C})$ are injections for all i. Thus it suffices to prove our theorem only for Kawai coverings.

To investigate Kawai coverings, we introduced multicyclic coverings in [2]. Let Y be a non-singular projective variety, and suppose that there are line bundles $\mathscr{L}_1, \mathscr{L}_2, \cdots, \mathscr{L}_m, \mathscr{L}$ on Y and is a surjective homomorphism

$$h\colon \bigoplus_{i=1}^m \mathscr{L}_i^{\otimes r} \longrightarrow \mathscr{L}$$

with a certain positive integer r. Let $P=\mathbb{P}(\bigoplus_{i=1}^m \mathscr{L}_i)$, $Q=\mathbb{P}(\bigoplus_{i=1}^m \mathscr{L}_i^{\otimes r})$ be the projective space bundles over Y associated with the vector bundles $\bigoplus_{i=1}^m \mathscr{L}_i$, $\bigoplus_{i=1}^m \mathscr{L}_i^{\otimes r}$ respectively. Then the homomorphism h defines a closed immersion $\sigma\colon Y\to Q$, while the natural inclusion $\bigoplus_{i=1}^m \mathscr{L}_i^{\otimes r}\to S^r(\bigoplus_{i=1}^m \mathscr{L}_i)$ induces a finite morphism $\Psi\colon P\to Q$ over Y. Let Z be the image of Y by σ and put $X=\Psi^{-1}(Z)$. Then we have the following commutative diagram:

$$(1.1)\qquad \begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "\psi"'] & P \arrow[d, "\Psi"] \arrow[dr, "pr_1"] & \\ Z \arrow[r, "i_Z"'] & Q \arrow[r, "pr_2"'] & Y \end{tikzcd}$$

and call the finite morphism $f=pr_1\circ i_X$ the *multicyclic covering* of Y associated with h. Let $\mathcal{O}_P(1)$ be the tautological line bundle of P and put $\mathcal{O}_X(t)=\mathcal{O}_P(t)\otimes\mathcal{O}_X$ for each $t\in\mathbb{Z}$. Then one has

$$(1.2)\qquad f_*\mathcal{O}_X(t)\cong\bigoplus_{(d_1,d_2,\cdots,d_m;d)\in I_r(t)}\mathscr{L}_1^{\otimes d_1}\otimes\mathscr{L}_2^{\otimes d_2}\otimes\cdots\otimes\mathscr{L}_m^{\otimes d_m}\otimes\mathscr{L}^{\otimes d},$$

where $I_r(t)=\{(d_1, d_2, \cdots, d_m; d)\in\mathbb{Z}^{m+1}\mid d_1+d_2+\cdots+d_m+rd=t;\ 0\leqq d_i<r \text{ for } i=1, 2, \cdots, m\}$ (cf. [2]). The homomorphism h induces homomorphisms $h_i\colon \mathscr{L}_i^{\otimes r}\to\mathscr{L}$ $(i=1, 2, \cdots, m)$ of line bundles. Every nonzero homomorphism h_i defines an effective divisor D_i whose ideal sheaf is $\mathscr{L}_i^{\otimes r}\otimes\mathscr{L}^*$. We call the homomorphism h a *regular quotient* of $\bigoplus_{i=1}^m \mathscr{L}_i$ with multiplicity r, if:

i) none of h_i's is a zero-homomorphism, and none or only one of h_i's is an isomorphism;

ii) each divisor D_i is non-singular, and the divisor $D_1+D_2+\cdots+D_m$ has only normal crossings.

If the homomorphism h is a regular quotient, then the covering space X is also a non-singular projective variety (cf. [2]). The homomorphism h is called *k-ample* if every nonzero divisor D_i is k-ample, that is to say, every nontrivial line bundle $\mathcal{O}(D_i)$ $(=\mathcal{L}_i^{\otimes(-r)}\otimes\mathcal{L})$ is k-ample in Sommese's sense (cf. [10]).

Now we return to reduction of the problem. Clearly we may assume the number of the irreducible components of the hypersurface S is larger than the dimension of $\mathbb{P}^n$. In this case, it has been shown in [2] that each Kawai covering can be factorized into a sequence of the multicyclic coverings associated with ample (0-ample) regular quotients:

$$U_v = X_{m+1} \longrightarrow X_m \longrightarrow X_{m-1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 = \mathbb{P}^n .$$

Consequently, we have only to prove the following Barth-type lemma for multicyclic coverings:

LEMMA. *Let Y be a non-singular n-dimensional projective variety and let $f\colon X \to Y$ be the multicyclic covering associated with a k-ample regular quotient on Y. Then the induced homomorphisms*

$$f^*\colon H^q(Y, \Omega_Y^p) \longrightarrow H^q(X, \Omega_X^p)$$

are isomorphisms for $|p+q-n|>k$.

On our problem, a Barth-type assertion arises in relation to the k-ampleness of the branch loci of coverings. Therefore it does not appear in our theorem in its explicit form.

§ 2. Barth-type theorem for multicyclic coverings

We prove Lemma in § 1. We assume that the covering $f\colon X \to Y$ is the multicyclic covering associated with a k-ample regular quotient $h\colon \bigoplus_{i=1}^m \mathcal{L}_i^{\otimes r} \to L$ of $\bigoplus_{i=1}^m \mathcal{L}_i$ with multiplicity r on Y and consider the diagram (1.1). First we prove the following:

PROPOSITION 1. *In the above situation, the homomorphisms*

$$H^q(X, f^*\Omega_Y^p \otimes \Lambda^t \Psi^* \Omega_{Q/Y}) \longrightarrow H^q(X, f^*\Omega_Y^p \otimes \Lambda^t \Omega_{P/Y})$$

induced by Ψ are isomorphisms for $p+q<n-k$ and $t \geqq 0$.

Proof. The natural homomorphism $\bigoplus_{i=1}^m f^*\mathcal{L}_i \otimes \mathcal{O}_X(-1) \to \mathcal{O}_X$ obtained from the universal quotient $\bigoplus_{i=1}^m pr_1^*\mathcal{L}_i \to \mathcal{O}_P(1)$ of P induces injective homomorphisms $f^*\mathcal{L}_i \otimes \mathcal{O}_X(-1) \to \mathcal{O}_X$ $(i=1, 2, \cdots, m)$ of line bundles. Tensoring each of these homomorphisms $r-1$ times with itself, we get a homomorphism $f^*\mathcal{L}_i^{\otimes(r-1)} \otimes \mathcal{O}_X(1-r) \to \mathcal{O}_X$. Tensor this with the line bundle $f^*\mathcal{L}_i \otimes \mathcal{O}_X(-1)$. Then one obtains a homomorphism $\theta_i\colon f^*\mathcal{L}_i^{\otimes r} \otimes \mathcal{O}_X(-r) \to f^*\mathcal{L}_i \otimes \mathcal{O}_X(-1)$. We define a homomorphism θ: $\bigoplus_{i=1}^m f^*\mathcal{L}_i^{\otimes r} \otimes \mathcal{O}_X(-r) \to \bigoplus_{i=1}^m f^*\mathcal{L}_i \otimes \mathcal{O}_X(-1)$ of vector bundles to be the direct sum of these $\{\theta_i\}$. Let $\mathcal{O}_Q(1)$ be the tautological line bundle of Q. Then by the definition of the morphism Ψ, one has $\Psi^*\mathcal{O}_Q(1) \cong \mathcal{O}_P(r)$. Hence from the well-known exact sequences on P and Q, we obtain the following commutative diagram on X

with exact rows:

$$(2.1)\qquad \begin{tikzcd} 0 \arrow[r] & \Psi^*\Omega_{Q/Y}\otimes\mathcal{O}_X \arrow[r, "\rho"] \arrow[d, "\tau"] & \bigoplus_{i=1}^m f^*\mathscr{L}_i^{\otimes r}\otimes\mathcal{O}_X(-r) \arrow[r] \arrow[d, "\theta"] & \mathcal{O}_X \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & \Omega_{P/Y}\otimes\mathcal{O}_X \arrow[r] & \bigoplus_{i=1}^m f^*\mathscr{L}_i\otimes\mathcal{O}_X(-1) \arrow[r] & \mathcal{O}_X \arrow[r] & 0, \end{tikzcd}$$

where τ is the homomorphism induced by Ψ and ρ is the usual homomorphism multiplied by the constant r (To check the commutativity, see Theorem 8.13 in [5]). From this we get commutative diagrams with exact rows:

$$(2.2)\qquad \begin{tikzcd} 0 \arrow[r] & \Lambda^t\Psi^*\Omega_{Q/Y}\otimes\mathcal{O}_X \arrow[r] \arrow[d] & \Lambda^t(\bigoplus_{i=1}^m f^*\mathscr{L}_i^{\otimes r}\otimes\mathcal{O}_X(-r)) \arrow[d, "\Lambda^t\theta"] \\ 0 \arrow[r] & \Lambda^t\Omega_{P/Y}\otimes\mathcal{O}_X \arrow[r] & \Lambda^t(\bigoplus_{i=1}^m f^*\mathscr{L}_i\otimes\mathcal{O}_X(-1)) \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r] & \Lambda^{t-1}\Psi^*\Omega_{Q/Y}\otimes\mathcal{O}_X \arrow[r] \arrow[d] & 0 \\ {} \arrow[r] & \Lambda^{t-1}\Omega_{P/Y}\otimes\mathcal{O}_X \arrow[r] & 0 \end{tikzcd}$$

($t=1, 2, \cdots, m-1$). Tensoring (2.2) with $f^*\Omega_Y^p$ and taking cohomology, we obtain the following commutative diagrams with long exact sequences:

(2.3)

$$\begin{tikzcd} \cdots \arrow[r] & H^q(X, f^*\Omega_Y^p\otimes\Lambda^t\Psi^*\Omega_{Q/Y}) \arrow[r] \arrow[d] & H^q(X, f^*\Omega_Y^p\otimes\Lambda^t(\bigoplus_{i=1}^m f^*\mathscr{L}_i^{\otimes r}\otimes\mathcal{O}_X(-r))) \arrow[d, "(\Lambda^t\theta)^q"] \\ \cdots \arrow[r] & H^q(X, f^*\Omega_Y^p\otimes\Lambda^t\Omega_{P/Y}) \arrow[r] & H^q(X, f^*\Omega_Y^p\otimes\Lambda^t(\bigoplus_{i=1}^m f^*\mathscr{L}_i\otimes\mathcal{O}_X(-1))) \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r] & H^q(X, f^*\Omega_Y^p\otimes\Lambda^{t-1}\Psi^*\Omega_{Q/Y}) \arrow[r] \arrow[d] & \cdots \\ {} \arrow[r] & H^q(X, f^*\Omega_Y^p\otimes\Lambda^{t-1}\Omega_{P/Y}) \arrow[r] & \cdots \end{tikzcd}$$

($t=1, 2, \cdots, m-1$). Now consider the homomorphism $\Lambda^t\theta$ in (2.2). This consists of the injective homomorphisms

$$\theta_{i_1}\otimes\theta_{i_2}\otimes\cdots\otimes\theta_{i_t}: f^*(\mathscr{L}_{i_1}^{\otimes r}\otimes\mathscr{L}_{i_2}^{\otimes r}\otimes\cdots\otimes\mathscr{L}_{i_t}^{\otimes r})\otimes\mathcal{O}_X(-rt)$$
$$\longrightarrow f^*(\mathscr{L}_{i_1}\otimes\mathscr{L}_{i_2}\otimes\cdots\otimes\mathscr{L}_{i_t})\otimes\mathcal{O}_X(-t)$$

($1\leq i_1<i_2<\cdots<i_t\leq m$). If $f^*(\mathscr{L}_{i_1}\otimes\mathscr{L}_{i_2}\otimes\cdots\otimes\mathscr{L}_{i_t})\otimes\mathcal{O}_X(-t)\cong\mathcal{O}_X$, then the homomorphism $\theta_{i_1}\otimes\theta_{i_2}\otimes\cdots\otimes\theta_{i_t}$ is an isomorphism. If not, by (1, 2) in § 1 and (1.10) in [10] one can check that the dual bundle of the vector bundle $\mathscr{L}_{i_1}\otimes\mathscr{L}_{i_2}\otimes\cdots\otimes\mathscr{L}_{i_t}\otimes f_*\mathcal{O}_X(-t)$ is a direct sum of k-ample line bundles. Hence we have

$$H^q(X, f^*\Omega_Y^p\otimes f^*(\mathscr{L}_{i_1}\otimes\mathscr{L}_{i_2}\otimes\cdots\otimes\mathscr{L}_{i_t})\otimes\mathcal{O}_X(-t))$$
$$\cong H^q(Y, \Omega_Y^p\otimes\mathscr{L}_{i_1}\otimes\mathscr{L}_{i_2}\otimes\cdots\otimes\mathscr{L}_{i_t}\otimes f_*\mathcal{O}_X(-t))\cong\{0\}$$

for $p+q<n-k$ (cf. (1.12) in [10]). By the same reason we have

$$H^q(X, f^*\Omega_Y^p \otimes f^*(\mathscr{L}_{i_1}^{\otimes r} \otimes \mathscr{L}_{i_2}^{\otimes r} \otimes \cdots \otimes \mathscr{L}_{i_t}^{\otimes r}) \otimes \mathcal{O}_X(-rt)) \cong \{0\}$$

$$\text{for} \quad p+q<n-k\,.$$

Thus the homomorphisms $(\Lambda^t\theta)^q$ in (2.3) are isomorphisms for $p+q<n-k$ and $1 \leqq t \leqq m-1$. Hence applying the Five Lemma to (2.3) and using induction for t, we obtain the result. QED

PROPOSITION 2. *The homomorphism*

$$H^q(X, \Psi^*\Omega_Q^p \otimes \mathcal{O}_X) \longrightarrow H^q(X, \Omega_P^p \otimes \mathcal{O}_X)$$

induced by Ψ is an isomorphism if $p+q<n-k$, and is an injection if $p+q=n-k$.

Proof. We have the following commutative diagram with exact rows:

$$(2.4)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & f^*\Omega_Y^1 & \longrightarrow & \Psi^*\Omega_Q^1 \otimes \mathcal{O}_X & \longrightarrow & \Psi^*\Omega_{Q/Y} \otimes \mathcal{O}_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & f^*\Omega_Y^1 & \longrightarrow & \Omega_P^1 \otimes \mathcal{O}_X & \longrightarrow & \Omega_{P/Y}\mathcal{O}_X & \longrightarrow & 0\,. \end{array}$$

From this we get finite filtrations

$$\Psi^*\Omega_Q^p \otimes \mathcal{O}_X = \mathscr{F}_p \supseteqq \mathscr{F}_{p-1} \supseteqq \cdots \supseteqq \mathscr{F}_1 \supseteqq \mathscr{F}_0 = f^*\Omega_Y^p\,,$$

$$\Omega_P^p \otimes \mathcal{O}_X = \mathscr{G}_p \supseteqq \mathscr{G}_{p-1} \supseteqq \cdots \supseteqq \mathscr{G}_1 \supseteqq \mathscr{G}_0 = f^*\Omega_Y^p$$

with the following commutative diagrams with exact rows:

$$(2.5)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F}_{j-1} & \longrightarrow & \mathscr{F}_j & \longrightarrow & f^*\Omega_Y^{p-j} \otimes \Lambda^j\Psi^*\Omega_{Q/Y} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{G}_{j-1} & \longrightarrow & \mathscr{G}_j & \longrightarrow & f^*\Omega_Y^{p-j} \otimes \Lambda^j\Omega_{P/Y} & \longrightarrow & 0 \end{array}$$

$(j=1, 2, \cdots, p)$. Then by taking cohomology, applying the Five Lemma and Proposition 1, and using induction for j, it follows from (2.5) that for every j the homomorphism $H^q(X, \mathscr{F}_j) \to H^q(X, \mathscr{G}_j)$ is an isomorphism if $p+q<n-k$, and is an injection if $p+q=n-k$. QED.

PROPOSITION 3. *Let $\mathscr{N}_X$ be the normal bundle of X in P. If the homomorphisms*

$$H^q(X, f^*\Omega_Y^p) \longrightarrow H^q(X, \Omega_X^p)$$

induced by f are isomorphisms for $p+q<s$ with some $s \leqq n-k$, the natural homomorphisms

$$H^q(X, \Lambda^t\mathscr{N}_X^* \otimes f^*\Omega_Y^p) \longrightarrow H^q(X, \Lambda^t\mathscr{N}_X^* \otimes \Omega_X^p)$$

are isomorphisms for $p+q<s$ and $t \geqq 0$.

Proof. Let $\mathscr{N}_Z$ be the normal bundle of Z in Q. Since $X=\Psi^{-1}(Z)$, we find that

$\psi^*\mathcal{N}_Z^* \cong \mathcal{N}_X^*$. Since Z is a section of Q over Y, we have $\mathcal{N}_Z^* \cong \Omega_{Q/Y} \otimes \mathcal{O}_Z$. Thus from the exact sequence $0 \to \Omega_{Q/Y} \to \bigoplus_{i=1}^m pr_2^* \mathcal{L}_i^{\otimes r} \otimes \mathcal{O}_Q(-1) \to \mathcal{O}_Q \to 0$ on Q, we get an exact sequence on X:

$$0 \longrightarrow \mathcal{N}_X^* \longrightarrow \bigoplus_{i=1}^m f^*\mathcal{L}_i^{\otimes r} \otimes \mathcal{O}_X(-r) \longrightarrow \mathcal{O}_X \longrightarrow 0\,. \tag{2.6}$$

From this we obtain the following commutative diagrams with exact rows:

$$\begin{array}{ccccc} 0 \longrightarrow \Lambda^t\mathcal{N}_X^* \otimes f^*\Omega_Y^p & \longrightarrow & \Lambda^t(\bigoplus_{i=1}^m f^*\mathcal{L}_i^{\otimes r} \otimes \mathcal{O}_X(-r)) \otimes f^*\Omega_Y^p \\ \downarrow & & \downarrow \\ 0 \longrightarrow \Lambda^t\mathcal{N}_X^* \otimes \Omega_X^p & \longrightarrow & \Lambda^t(\bigoplus_{i=1}^m f^*\mathcal{L}_i^{\otimes r} \otimes \mathcal{O}_X(-r)) \otimes \Omega_X^p \end{array} \tag{2.7}$$

$$\begin{array}{c} \longrightarrow \Lambda^{t-1}\mathcal{N}_X^* \otimes f^*\Omega_Y^p \longrightarrow 0 \\ \downarrow \\ \longrightarrow \Lambda^{t-1}\mathcal{N}_X^* \otimes \Omega_X^p \longrightarrow 0 \end{array}$$

$(t = 1, 2, \cdots, m-1)$. Since $f^*\mathcal{L} \cong \mathcal{O}_X(r)$, we have

$$f^*(\mathcal{L}_{i_1}^{\otimes r} \otimes \mathcal{L}_{i_2}^{\otimes r} \otimes \cdots \otimes \mathcal{L}_{i_t}^{\otimes r}) \otimes \mathcal{O}_X(-rt) \cong f^*(\mathcal{L}_{i_1}^{\otimes r} \otimes \mathcal{L}_{i_2}^{\otimes r} \otimes \cdots \otimes \mathcal{L}_{i_t}^{\otimes r} \otimes \mathcal{L}^{\otimes(-t)})$$
$$(1 \leqq i_1 < i_2 < \cdots < i_t \leqq m)\,.$$

Therefore the dual line bundle of $f^*(\mathcal{L}_{i_1}^{\otimes r} \otimes \mathcal{L}_{i_2}^{\otimes r} \otimes \cdots \otimes \mathcal{L}_{i_t}^{\otimes r}) \otimes \mathcal{O}_X(-rt)$ is k-ample unless it is isomorphic to $\mathcal{O}_X$ (cf. (1.8) in [10]). Hence in the same manner as in Proof of Proposition 1, we find that the homomorphisms

$$H^q(X, \Lambda^t(\bigoplus_{i=1}^m f^*\mathcal{L}_i^{\otimes r} \otimes \mathcal{O}_X(-r)) \otimes f^*\Omega_Y^p)$$
$$\longrightarrow H^q(X, \Lambda^t(\bigoplus_{i=1}^m f^*\mathcal{L}_i^{\otimes r} \otimes \mathcal{O}_X(-r)) \otimes \Omega_X^p)$$

are isomorphisms for $p+q<n-k$ and for every t, and furthermore obtain the result. QED

Proof of Lemma. Since $\psi^*\Omega_Z^1 \cong f^*\Omega_Y^1$ and $\psi^*\mathcal{N}_Z \cong \mathcal{N}_X$, we have the following commutative diagram on X with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N}_X^* & \longrightarrow & \Psi^*\Omega_Q^1 \otimes \mathcal{O}_X & \longrightarrow & f^*\Omega_Y^1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{N}_X^* & \longrightarrow & \Omega_P^1 \otimes \mathcal{O}_X & \longrightarrow & \Omega_X^1 & \longrightarrow & 0\,. \end{array} \tag{2.8}$$

For each $1 \leqq p \leqq n$, we get from (2.8) finite filtrations

$$\Psi^*\Omega_Q^p \otimes \mathcal{O}_X = \mathcal{F}_p^p \supseteqq \mathcal{F}_{p-1}^p \supseteqq \cdots \supseteqq \mathcal{F}_1^p \supseteqq \mathcal{F}_0^p = \Lambda^p\mathcal{N}^*{}_X\,,$$
$$\Omega_P^p \otimes \mathcal{O}_X = \mathcal{G}_p^p \supseteqq \mathcal{G}_{p-1}^p \supseteqq \cdots \supseteqq \mathcal{G}_1^p \supseteqq \mathcal{G}_0^p = \Lambda^p\mathcal{N}_X^*$$

with commutative diagrams with exact rows:

$$
(2.9)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{F}^p_{j-1} & \longrightarrow & \mathscr{F}^p_j & \longrightarrow & \Lambda^{p-j}\mathscr{N}^*_X\otimes f^*\Omega^j_Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{G}^p_{j-1} & \longrightarrow & \mathscr{G}^p_j & \longrightarrow & \Lambda^{p-j}\mathscr{N}^*_X\otimes\Omega^j_X & \longrightarrow & 0
\end{array}
$$

$(j=1,2,\cdots,p)$. Then taking cohomology, applying the Five Lemma, Proposition 2 and 3, and using induction for p and j, we see that the homomorphisms $H^q(X,\mathscr{F}^p_j)\rightarrow H^q(X,\mathscr{G}^p_j)$ are isomorphisms for $j+q<n-k$, and that the homomorphisms $H^q(X,f^*\Omega^p_Y)\rightarrow H^q(X,\Omega^p_X)$ are isomorphisms for $p+q<n-k$. On the other hand, we have $H^q(X,f^*\Omega^p_Y)\cong H^q(Y,\Omega^p_Y\otimes f_*\mathcal{O}_X)$ for every p, q. The dual line bundle of $\mathscr{L}_1^{\otimes d_1}\otimes\mathscr{L}_2^{\otimes d_2}\otimes\cdots\otimes\mathscr{L}_m^{\otimes d_m}\otimes\mathscr{L}^{\otimes d}$ is k-ample if $(d_1,d_2,\cdots,d_m;d)\in I_r(0)-\{(0,0,\cdots,0;0)\}$. Thus it follows from (1.2) in §1 and (1.12) in [10] that $H^q(Y,\Omega^p_Y\otimes f_*\mathcal{O}_X)\cong H^q(Y,\Omega^p_Y)$ for $p+q<n-k$. Finally note that the homomorphisms $H^q(Y,\Omega^p_Y)\rightarrow H^q(X,\Omega^p_X)$ are injections for every p, q (cf. Satz 7 in [4]). Hence using the Serre duality theorem, we obtain the desired result. QED

Acknowledgements. The author would like to express his thanks to Prof. S. Kawai for many helpful suggestions and continuous encouragement.

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